8.4 Nonhomogeous Problems Green's Formula

$$\text{IBVP}\begin{cases} u_t = k u_{xx} + Q(x,t) & (1.1) \\ u(0,t) = A(t), \quad u(L,t) = B(t) & (1.2) \\ u(x,0) = f(x) & (1.3) \end{cases}$$

<u>Idea:</u>

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} b_n'(t)\, \phi_n(x)$$

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} d_n(t)\, \phi_n(x)$$

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\, \phi_n(x).$$

then, Subst. into (1.1) ...

$$\boxed{\sum_{n=1}^{\infty} \left[b_n'(t) - k\, d_n(t) - q_n(t) \right] \phi_n(x) = 0}$$

where $\phi_n(x)$ satisfies:

$$\begin{cases} \phi_n''(x) + \lambda_n \phi_n(x) = 0 \\ \phi_n(L) = 0, \quad \phi_n(0) = 0 \end{cases} \qquad (1.4)$$

Then,

$$\boxed{b_n'(t) = k\, d_n(t) + q_n(t),} \quad n=1,2\ldots \qquad (1.5)$$

?

known.

Notice, $d_n(t)$ Can be written in terms of $b_n(t)$ using Green's formula.

If this is possible, then we end up with an initial value problem for $b_n(t)$ using the I.C.

$$u(x,0) = \sum_{n=1}^{\infty} b_n(0)\,\phi_n(x) = f(x)$$

$$= \sum_{n=1}^{\infty} f_n\,\phi_n(x)$$

$$\Rightarrow \boxed{b_n(0) = f_n, \quad n=1,2\ldots} \qquad (1.6)$$

IVP for $b_n(t)$ (1.5)-(1.6).

Therefore, using that $\phi_n'' = -\lambda_n \phi_n$

$$\boxed{\int_0^L u_{xx}(x,t)\,\phi_n(x)\,dx = -\lambda_n \int_0^L u(x,t)\,\phi_n(x)\,dx + A(t)\,\phi_n'(0) - B(t)\,\phi_n'(L).} \tag{4.1}$$

Substituting (4.1) into (3.1) yields

$$a_n(t) = -\lambda_n \frac{\int_0^L u(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} + \frac{A(t)\,\phi_n'(0) - B(t)\,\phi_n'(L)}{\int_0^L \phi_n^2(x)\,dx}.$$

or
$$\boxed{a_n(t) = -\lambda_n b_n(t) + \frac{A(t)\,\phi_n'(0) - B(t)\,\phi_n'(L)}{\int_0^L \phi_n^2(x)\,dx}.}$$

Finally, Substitution into (1.1)

$$\boxed{\begin{aligned} b_n'(t) + \lambda_n b_n(t) &= \underbrace{\frac{A(t)\,\phi_n'(0) - B(t)\,\phi_n'(L)}{\int_0^L \phi_n^2(x)\,dx}}_{= G_n(t)} + q_n(t) \\ b_n(0) &= f_n \end{aligned}}$$

Linear 1st Order ODE.

Heat conduction with time-dep. boundary condition and source function.

IBVP
$$\begin{cases} u_t = k u_{xx} + \cos t, & 0 < x < L \quad (1.1) \\ u(0,t) = t, \quad u(L,t) = 0 & (1.2) \\ u(x,0) = f(x) & (1.3) \end{cases}$$

Answ: We will use Green's formula.

The associated homogeneous problem is

$$\begin{cases} V_t = k V_{xx} \\ V(0,t) = 0, \quad V(L,t) = 0 \\ V(x,0) = f(x) \end{cases}$$

Using sep. of Vabs, we arrive to the eigenvalue problem

$$\begin{cases} \phi''(x) + \lambda \phi(x) = 0 \\ \phi(0) = 0, \quad \phi(L) = 0 \end{cases}$$

whose eigenvalues are $\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$ (1.4)

and corresponding eigenfunctions are

$$\boxed{\phi_n(x) = \sin\left(\frac{n\pi}{L} x\right)}, \quad n = 1, 2 \ldots \quad (1.5)$$

Since $\phi_n(x)$ form a complete set, then

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)}, \quad \text{for } t \text{ fixed} \qquad (2.1)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x). \qquad (2.2)$$

$$\boxed{u_{xx}(x,t) = \sum_{n=1}^{\infty} d_n(t)\,\phi_n(x)} \qquad (2.3)$$

Why we cannot perform term-term differentiation for this IBVP. ?

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x). \qquad (2.4)$$

Subst. of (2.1) – (2.4) into (1.1) leads to

$$\sum_{n=1}^{\infty} \left[a_n'(t) - k\,d_n(t) - q_n(t) \right] \phi_n(x) = 0$$

$$\Rightarrow \quad \boxed{a_n'(t) - k\,d_n(t) = q_n(t), \quad n = 1, 2.} \ldots \qquad (2.5)$$

First order ODE if we can express $d_n(t)$ in terms of $a_n(t)$.

Using I.C.

$$u(x,0) = \sum_{n=1}^{\infty} a_n(0)\,\phi_n(x) = f(x)$$

$$= \sum_{n=1}^{\infty} f_n\,\phi_n(x)$$

then, we arrive to an IC. for (2.5)

$$\boxed{b_n(0) = f_n,} \quad n = 1, 2 \ldots$$

Notice that using orthog. of ϕ_n in (2.3)

We arrive

$$\boxed{d_n(t) = \frac{\int_0^L u_{xx}(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}} \quad (2.6)$$

Then, applying Green's formula (Why?)

$$\boxed{\int_0^L [v''w - w''v]\,dv = v'w - w'v\Big|_0^L} \quad (2.7)$$

and calling $v = u(x,t)$, $w = \phi_n(x)$

$$\int_0^L u_{xx}(x,t)\,\phi_n(x)\,dx = \int_0^L \phi_n''(x)\,u(x,t)\,dx$$

$$\Big(\phi_n(x)\,u_x(x,t) - u(x,t)\,\phi_n'(x)\Big)\Big|_0^L$$

$$= \overset{=0}{\phi_n(L)}\,u_x(L,t) - \overset{=0}{u(L,t)}\,\phi_n'(L)$$

$$-\Big(\overset{=0}{\phi_n(0)}\,u_x(0,t) - \overset{=t}{u(0,t)}\,\phi_n'(0)\Big)$$

$$= t\,\phi_n'(0)$$

Using $\phi_n''(x) = -\lambda_n \phi_n(x)$.

$$\boxed{\int_0^L u_{xx}(x,t)\,\phi_n(x)\,dx = -\lambda_n \int_0^L u(x,t)\,\phi_n(x)\,dx + t\,\phi_n'(0)} \quad (4.1)$$

Subst. into (2.6) of (4.1)

$$a_n(t) = -\lambda_n \frac{\int_0^1 u(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} + \overbrace{\frac{t\,\phi_n'(0)}{\int_0^L \phi_n^2(x)\,dx}}^{=\alpha t}$$

or

$$\boxed{a_n(t) = -\lambda_n\,a_n(t) + \alpha t} \quad (4.2)$$

$n=1,2\ldots$

Subst. into (2.5)

$$\boxed{a_n'(t) + \lambda_n k\, a_n(t) = q_n(t) + \alpha t} \qquad (5.1)$$

Also

$$\boxed{a_n(0) = f_n} \qquad (5.2)$$

(5.1)-(5.2) is an IVP for a 1^{st} order ODE

It can be solved by introducing int. factor

$$\left[a_n(t)\, e^{\lambda_n k t}\right]' = \left(q_n(t) + \alpha t\right) e^{\lambda_n k t}$$

$$\int_0^t \qquad a_n(t)\, e^{\lambda_n k t} = \int_0^t \left(q_n(\bar{t}) + \alpha \bar{t}\right) e^{\lambda_n k \bar{t}}\, d\bar{t}$$

IPP.

or

$$\boxed{a_n(t) = \frac{1}{\lambda_n k t} \int_0^t \left(\cos \bar{t} + 2\bar{t}\right) e^{\lambda_n k \bar{t}}\, d\bar{t}} \qquad (5.3)$$

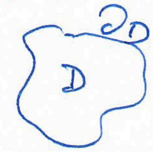
Finally, the soln. of our original problem is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi}{L} x\right)} \qquad (5.4)$$

where $a_n(t)$ is given by (5.3).

My Membrane Vibration problem:

IBVP
$$\begin{cases} u_{tt} = c^2 \nabla^2 u + \cos(3t), & \text{on } D. \\ u = 0, & \text{on } \partial D. \\ u(x,y,0) = \alpha(x,y) & \\ u_t(x,y,0) = \beta(x,y) & \end{cases}$$



Answ: we will use